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Characteristic Classes of Piecewise Differentiable Affine Connections on Smooth Manifolds*

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We show that the Chern–Weil construction can still be used to extract the characteristic classes of smooth manifolds even from (discontinuous) piecewise differentiable Riemannian metrics and affine connections.

1. Introduction

In [T] we defined and studied the algebra of generalized piecewise differentiable distributions and currents on smooth manifolds.

This paper constitutes a first application of those results.

We show here that the Chern–Weil construction for the characteristic classes can be extended to the case of piecewise differentiable Riemannian metrics. This paves the way to a general framework for attacking the problem of describing local combinatorial invariants on smooth manifolds.

2. The Complex of Symmetric Generalized Piecewise Differential Currents

Let M be an oriented smooth manifold and let T be a smooth triangulation of M. For any $k \in \mathbb{N}$, we set $M_k \doteq M \times \cdots \times M$ (k factors); $T^k \doteq T \times \cdots \times T$ gives a decomposition of M_k by polysimplices.

In [T] we defined the differential complex of generalized piecewise differentiable (g.p.d.) currents, $\mathscr{C}^*(M)$.

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The wedge product $\wedge$ in $\mathscr{C}^*_T(M)$ is defined by means of the natural homomorphisms

$$\mu\colon \mathscr{C}^*(T^{k_1}) \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} \mathscr{C}^*(T^{k_r}) \to \mathscr{C}^*(T^{k_1 + \cdots + k_r});$$

see [T, Sect. 4].

The wedge product is not graded commutative.

In some applications it might be desirable to have a commutative product defined.

To this purpose, we will discuss first a commutative product for distributions.

The group of permutations S_k acts naturally on M_k as a group of diffeomorphisms; S_k induces an action on the space of p.d. distributions on M_k and this action passes to the quotient space $\mathscr{C}^0(T^k)$ of g.p.d. distributions.

Let $\mathscr{A}$ denote averaging on S_k:

$$\mathscr{A}(\ ) = \frac{1}{k!} \sum_{\sigma \in S_k} \sigma(\ ).$$

The space $\mathscr{A}(\mathscr{C}^0(T^k))$, which consists precisely of those elements of $\mathscr{C}^0(T^k)$ which are kept fixed by the group of permutations S_k, *can be described also as a* quotient space

$$S\mathscr{C}^0(T^k) \doteq \mathscr{C}^0(T^k)/W, \tag{2.1}$$

where W is the vector subspace generated by all elements of the form $\mathsf{U} - \sigma(\mathsf{U})$, where $\mathsf{U} \in \mathscr{C}^0(T^k)$, and $\sigma \in S_k$.

The mapping

$$S\mathscr{C}^0(T^k) \to S\mathscr{C}^0(T^{k+1}), \qquad k \in \mathbb{N},$$
$$[\mathsf{U}] \mapsto [\mathsf{U} \otimes 1]$$

defines an inductive system; we introduce

$$S\mathscr{C}^0_T(M) \doteq \varinjlim_k S\mathscr{C}^0(T^k).$$

An element of $S\mathscr{C}^0(T^k)$ will be called a *symmetric* g.p.d. distribution on M. This space depends on the triangulation T; by taking the direct limit

$$S\mathscr{C}^0(M) \doteq \varinjlim_T S\mathscr{C}^0_T(M), \tag{2.2}$$

we obtain a space of generalized distributions, independent of any particular triangulation of M. We define also

$$\mathscr{C}^0(M) \doteq \varinjlim_T \mathscr{C}^0_T(M). \tag{2.2$'$}$$

2.1. *Remark.* For cohomological considerations, we shall require from the triangulation T to verify Axiom C.2 from [T].

The projection mapping

$$\pi_{T,k}: \mathscr{C}^0(T^k) \to S\mathscr{C}^0(T^k) \tag{2.3}$$

defines a homomorphism

$$\pi: \varinjlim_{k,T} \mathscr{C}^0(T^k) \to \varinjlim_{k,T} S\mathscr{C}^0(T^k). \tag{2.4}$$

If $\mathsf{U}_1, ..., \mathsf{U}_h \in \mathscr{C}^0(M)$, let $\mathsf{U}_1, ..., \mathsf{U}_h \in \mathscr{C}^0(M)$ denote their product as defined in [T, Sect. 4].

By definition,

$$\mathsf{U}_1 \circ \cdots \circ \mathsf{U}_h = \pi(\mathsf{U}_1 \cdots \mathsf{U}_h) \tag{2.5}$$

will be called the *symmetric product* of $\mathsf{U}_1, ..., \mathsf{U}_h$.

The symmetric product is associative and commutative.

If σ and τ are two g.p.d. currents on M, they can be described locally as

$$\sigma = \sigma_{i_1 \cdots i_h} dx^{i_1} \wedge \cdots \wedge dx^{i_h},$$

$$\tau = \tau_{j_1 \cdots j_l} dx^{j_1} \wedge \cdots \wedge dx^{j_l},$$

where $\sigma_{i_1 \cdots i_h}$, $\tau_{j_1 \cdots j_l}$ are g.p.d. distributions.

By definition, their symmetric *wedge product*, $\sigma \mathring{\wedge} \tau$, is

$$\sigma \mathring{\wedge} \tau = \sigma_{i_1 \cdots i_h} \circ \tau_{j_1 \cdots j_l} dx^{i_1} \wedge \cdots \wedge dx^{i_h} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_l}. \tag{2.6}$$

Let $S\mathscr{C}^*(M)$ denote the space of all g.p.d. currents which can be described locally as differential forms whose coefficients are symmetric g.p.d. distributions.

We have the

2.2. THEOREM. *$S\mathscr{C}^*(M)$ is a graded commutative differential algebra.*

2.3. THEOREM. *If the triangulations of T from* (2.2) *satisfy Axiom* C.2 *from* [T, Sect. 3], *then the natural chain homomorphism*

$$j: \Omega^*(M) \to S\mathscr{C}^*(M) \tag{2.7}$$

$j: \omega \mapsto \{\omega \otimes_{\mathbb{R}} 1 \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} 1\}$, *induces monomorphism in homology* (*here* $\{\ \}$ *denotes the class in* $S\mathscr{C}^*(M)$, *and* $\Omega^*(M)$ *denotes the deRham complex*).

Proof. It is sufficient to show that for any $k \in \mathbb{N}$, and for any T satisfying C.3, the inclusion

$$\Omega^*(M) \to S\mathscr{C}^*(T^k)$$

induces monomorphism in homology. Let ω be a closed smooth form on M, and suppose that

$$[j(\omega)] = 0 \in H_*(S\mathscr{C}^0(T^k)).$$

Let us denote by W^* the space of those currents which are represented by differential forms with coefficients in W (see (2.1)). Then there exist $\sigma \in \mathscr{C}^*(T^k)$ and $\theta \in W^*$ such that

$$j(\omega) = \{\omega \otimes_{\mathbb{R}} 1 \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} 1\} = d\sigma + \theta.$$

By averaging over the group S_k, we have in $\mathscr{C}^*(T^k)$, as $\mathscr{A}\theta = 0$,

$$\mathscr{A}(j(\omega)) = \mathscr{A}(d\sigma + \theta) = \mathscr{A}\, d\sigma = d\,\mathscr{A}\sigma,$$

which shows that

$$[\mathscr{A}j(\omega)] = 0 \in H_*(\mathscr{C}^*(T^k)). \tag{2.8}$$

But

$$\mathscr{A}j(\omega) = \frac{1}{k} \sum_{i=1}^{k} \{1 \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} 1 \otimes_{\mathbb{R}} \underset{(i)}{\omega} \otimes_{\mathbb{R}} 1 \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} 1\},$$

and because this is a smooth form

$$j(\mathscr{A}\omega) = j(\omega). \tag{2.9}$$

From (2.8), (2.9), and Theorem 4.3(iii) [T], it follows that $[\omega] = 0$ in the deRham cohomology.

3. Homotopy Invariance

With the same notations as above, we compare the homologies of the complexes $\mathscr{C}^*(M)$, resp. $S\mathscr{C}^*(M)$, and $\mathscr{C}^*(M \times \mathbb{R})$, resp. $S\mathscr{C}^*(M \times \mathbb{R})$.

For the application in Section 5 it will be sufficient to restrict our attention to a special class of decompositions of $M \times \mathbb{R}$.

For this purpose, if $T = \{\sigma_\alpha^{(r)}\}$ is a triangulation of M, we consider the decomposition

$$T \times \mathbb{R} \doteq \{\sigma_\alpha^{(r)} \times \mathbb{R}\}$$

of $M \times \mathbb{R}$, and we define

$$\begin{aligned} \hat{\mathscr{C}}^*(M \times \mathbb{R}) &\doteq \varinjlim_{T,k} \mathscr{C}^*((T \times \mathbb{R})^k) \\ S\hat{\mathscr{C}}^*(M \times \mathbb{R}) &\doteq \varinjlim_{T,k} S\mathscr{C}^*((T \times \mathbb{R})^k). \end{aligned} \tag{3.1}$$

The condition on the type of decomposition allowed in the definition of $\hat{\mathscr{C}}^*(M \times \mathbb{R})$ amounts to the requirement that all currents used for its construction are smooth in the direction of the factor $\mathbb{R}$.

This observation justifies us in speaking about the *restriction* of any g.p.d. current $\omega \in \hat{\mathscr{C}}^*(M \times \mathbb{R})$ to the level submanifold $M \times \{t\} \subset M \times \mathbb{R}$; this restriction will be denoted $r_t^* \omega$.

3.1. PROPOSITION. *The restriction homomorphism r_t^* induces isomorphism in homology*:

$$\begin{aligned} r_t^* &: H_*(\hat{\mathscr{C}}^*(M \times \mathbb{R})) \to H_*(\mathscr{C}^*(M)) \\ r_t^* &: H_*(S\hat{\mathscr{C}}^*(M \times \mathbb{R})) \to H_*(S\mathscr{C}^*(M)). \end{aligned} \tag{3.2}$$

Proof. As all g.p.d. currents defining $\hat{\mathscr{C}}^*(M \times \mathbb{R})$, resp. $S\hat{\mathscr{C}}^*(M \times \mathbb{R})$, are smooth in the $\mathbb{R}$-direction, the classical homotopy operators for the deRham cohomology can be used.

3.2. COROLLARY. *If γ is a cocycle in $\hat{\mathscr{C}}^*(M \times \mathbb{R})$, resp. $S\hat{\mathscr{C}}^*(M \times \mathbb{R})$, then*

$$[r_0^* \gamma] = [r_1^* \gamma] \in H_*(\mathscr{C}^*(M)), \text{ resp. } H_*(S\mathscr{C}^*(M)).$$

3.3. COROLLARY. *If γ is a cocycle in $\hat{\mathscr{C}}^*(M \times \mathbb{R})$, resp. $S\hat{\mathscr{C}}^*(M \times \mathbb{R})$, and if $r_0^* \gamma$ is a smooth differential form, then*

$$\left.\begin{array}{l} H_*(\mathscr{C}^*(M)) \\ \text{resp.} \\ H_*(S\mathscr{C}^*(M)) \end{array}\right\} \ni [r_1^* \gamma] = [r_0^* \gamma] \in H_{dR}^*(M),$$

where H_{dR}^* is the deRham cohomology, seen as a vector subspace of $H_*(\mathscr{C}^*(M))$, resp. $H_*(S\mathscr{C}^*(M))$.

4. GENERALIZED PIECEWISE DIFFERENTIABLE LINEAR CONNECTIONS; CURVATURE AND CHARACTERISTIC CLASSES

We may extend the previous constructions to sections in vector bundles. Indeed, for $\xi \to M$, a smooth vector bundle, we extend the space of *smooth*

r-differential forms on M with values in ξ, $\Omega^r(M, \xi)$ by allowing their coefficients to be (*symmetric*) *generalized piecewise differentiable distributions*, and by doing so, we obtain the spaces

$$\begin{aligned} \mathscr{C}^r(M, \xi) &\doteq \mathscr{C}^0(M) \otimes_{\Omega^0(M)} \Omega^r(M, \xi) \\ S\mathscr{C}^r(M, \xi) &\doteq S\mathscr{C}^0(M) \otimes_{\Omega^0(M)} \Omega^r(M, \xi). \end{aligned} \tag{4.1}$$

4.1. Definition. A *generalized piecewise differentiable linear connection* in the smooth vector bundle ξ is a linear homomorphism

$$\nabla \colon \mathscr{C}^0(M, \xi) \to \mathscr{C}^1(M, \xi),$$

resp.

$$\nabla \colon S\mathscr{C}^0(M, \xi) \to S\mathscr{C}^1(M, \xi),$$

which satisfies the Leibnitz formula

$$\begin{aligned} \nabla(f \circ s) = df \circ s + f \circ \nabla s, \qquad &\text{for any } f \in \mathscr{C}^0(M), \text{ resp. } S\mathscr{C}^0(M) \\ &\text{and any } s \in \mathscr{C}^0(M, \xi), \text{ resp. } S\mathscr{C}^0(M, \xi). \end{aligned} \tag{4.2}$$

From this point on there will be no difficulty in defining the *curvature* and the *characteristic classes* of the g.p.d. connection ∇ (see Connes [C], Karoubi [K], and Milnor [M]).

4.2. Theorem. (i) *For any smooth vector bundle $\xi \to M$, and for any generalized piecewise differentiable connection ∇ in ξ, the curvature of the connection, R_∇, and the g.p.d. current $\psi_k \doteq \mathrm{Trace}(R_\nabla)^k$, $k \in \mathbb{N}$, are defined.*

(ii) *The g.p.d. current ψ_k is closed, and its homology class, $[\psi_k] \in H_*(\mathscr{C}^*(M))$, resp. $H_*(S\mathscr{C}^*(M))$ is independent of the connection.*

(iii) *If all triangulations used in the definition of the g.p.d. currents satisfy Axiom* C.2, *then the class $[\psi_k]$ coincides with the same characteristic class* (*in the deRham cohomology*) *of a smooth connection in ξ.*

Proof. This follows from Corollary 3.3.

5. Piecewise Differentiable Riemannian Metrics; Curvature and Characteristic Classes

Let Γ be a piecewise differentiable Riemannian metric on M. We mean by this that M has a smooth triangulation and that on each closed top dimensional simplex of the triangulation, a smooth (up to the boundary) Riemannian metric is given.

In a smooth system of coordinates $x = (x^1, ..., x^n)$, the metric Γ is represented as

$$\Gamma = g_{ij}\, dx^i\, dx^j,$$

where g_{ij} are piecewise differentiable functions.

Let $\mathscr{U} = \{U_\alpha\}_{\alpha \in A}$ be a locally finite open covering of M by coordinate charts $x_\alpha = (x^1_\alpha, ..., x^\omega_\alpha)$, and let $\{\rho_\alpha\}_{\alpha \in A}$ be a partition of unity subordinate to $\mathscr{U}$.

For every $U_\alpha \in \mathscr{U}$, we define the Christoffel symbols

$$\Gamma^i_{(\alpha)jk} = \frac{1}{2}\, g^{ir} \circ \left\{ \frac{\partial g_{rj}}{\partial x^k} + \frac{\partial g_{rk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^r} \right\}, \tag{5.1}$$

which define a g.p.d. Levi-Civita connection on ∇_α on U_α by the formula

$$\nabla_\alpha(dx^i_\alpha) = \Gamma^i_{(\alpha)jk}\, dx^j_\alpha \otimes dx^k_\alpha. \tag{5.2}$$

The components g^{ir} in (5.1) are piecewise differentiable functions on U_α, computed pointwise.

The components $\Gamma^i_{(\alpha)jk}$ are g.p.d. distributions.

We can patch together the local connections ∇_α to define a global *g.p.d. Levi-Civita connection* ∇ *on* M:

$$\nabla(\omega) = \sum_{\alpha \in A} \nabla_\alpha(\rho_\alpha \omega) \tag{5.3}$$

for any differential form ω.

When Γ is smooth, the g.p.d. Levi-Civita connection ∇ coincides with the classical Levi-Civita connection of Γ.

Note, however, that the g.p.d. Levi-Civita connection ∇ depends upon the choice of the partition of unity and the coordinate functions. This is due to the fact that, in general,

$$g^{ir} \circ g_{rj} \neq \delta^i_j \qquad \text{in} \quad \mathscr{C}^0(U_\alpha). \tag{5.4}$$

From this point on we can apply the results from Section 4 to define the curvature of ∇ and the corresponding characteristic classes. We have

5.1. THEOREM. *Given a piecewise differentiable Riemannian metric* (*with respect to a triangulation which satisfies Axiom* C.2) *on a smooth manifold, the Chern–Weil construcion leads to a description of its characteristic classes expressed in terms of generalized piecewise differentiable currents.*

REFERENCES

[C] A. CONNES, Non-commutative differential geometry, *Inst. Hautes Études Sci. Publ. Math.* **62** (1986), 257–360.

[K] M. KAROUBI, "Homologie cyclique et *K*-théorie," Soc. Math. de France, 1987.

[M] J. MILNOR AND J. STASHEFF, "Characteristic Classes," Annals of Mathematics Studies, Vol. 76, Princeton Univ. Press, Princeton, NJ, 1974.

[T] N. TELEMAN, The algebra of piecewise differentiable currents on smooth manifolds, *J. Funct. Anal.* **99** (1991), 191–214.

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